

# A NOTE ON MAPPING CLASS GROUP ACTIONS ON DERIVED CATEGORIES.

NICOLÒ SIBILLA

ABSTRACT. Let  $X_n$  be a cycle of  $n$  projective lines, and  $\mathbb{T}_n$  a symplectic torus with  $n$  punctures. Using the theory of spherical twists introduced by Seidel and Thomas [ST], I will define an action of the pure mapping class group of  $\mathbb{T}_n$  on  $D^b(\text{Coh}(X_n))$ . The motivation comes from homological mirror symmetry for degenerate elliptic curves, which was studied by the author with Treumann and Zaslow in [STZ].

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## 1. INTRODUCTION.

According to Kontsevich's *Homological Mirror Symmetry* conjecture (from now on HMS, see [K]), given a Calabi-Yau variety  $X$  and a symplectic manifold  $\tilde{X}$ , if  $X$  and  $\tilde{X}$  are mirror partners, then the derived category of coherent sheaves over  $X$ ,  $D^b(\text{Coh}(X))$ , should be quasi-equivalent to the Fukaya category of  $\tilde{X}$ ,  $\text{Fuk}(\tilde{X})$ . Since  $\text{Fuk}(\tilde{X})$  is an invariant of the symplectic geometry of  $\tilde{X}$ , mirror symmetry predicts that the group of symplectic automorphisms of  $\tilde{X}$  acts by equivalences on  $D^b(\text{Coh}(X))$ . In [ST] Seidel and Thomas investigate this aspect of HMS by introducing the notions of *spherical object* and *twist functor*, which can be defined for general triangulated categories, and axiomatize the formal homological properties enjoyed by equivalences of the Fukaya category induced by *generalized Dehn twists* (these are special symplectic automorphisms introduced by Seidel, see [S]). Using their theory they are able, in many interesting examples, to give a conjectural description of the equivalences of  $D^b(\text{Coh}(X))$  which should be mirror to symplectic automorphisms of  $\tilde{X}$ . I refer the reader to [ST] for a detailed account of this circle of ideas. A brief overview of the relevant definitions will be given in Section 3 below.

Let  $X_n$  be a cycle of  $n$  projective lines, i.e. a nodal curve of arithmetic genus 1, with  $n$  singular points. Well known mirror symmetry heuristics suggest that the mirror of  $X_n$  should be a symplectic torus with  $n$  punctures, which I shall denote  $\mathbb{T}_n$ . In the paper [STZ], joint with Treumann and Zaslow, we prove a version of HMS for  $X_n$  and  $\mathbb{T}_n$ , by showing that the category of perfect complexes over  $X_n$ ,  $\text{Perf}(X_n)$ , is quasi-equivalent to a certain conjectural model of  $\text{Fuk}(\mathbb{T}_n)$  which we develop in the paper. See also the recent work [LP]

in which the authors prove, with very different techniques, a HMS statement for the case  $n = 1$ .

Motivated by [STZ], in this paper I explore the consequences of mirror symmetry for the study of auto-equivalences of  $D^b(\text{Coh}(X_n))$ . Recall that the mapping class group of an oriented surface  $\Sigma$  can be described as the group of symplectic automorphisms of  $\Sigma$ , modulo isotopy. The existence of an action of the mapping class group of  $\mathbb{T}_n$  on  $D^b(\text{Coh}(X_n))$  does not follow directly from [STZ], as the model of the Fukaya category considered there is not acted upon, in any obvious way, by symplectomorphisms of  $\mathbb{T}_n$ <sup>1</sup>. My main result uses the framework of [ST] to construct an action of the (pure) mapping class group of  $\mathbb{T}_n$ ,  $\text{PM}(\mathbb{T}_n)$ , over  $D^b(\text{Coh}(X_n))$ . In future work, I plan to establish that this action is faithful. It is worth pointing out that the action I will define is, in an appropriate sense, a categorification of the symplectic representation of the mapping class group, which can be recovered by considering the induced action on the *numerical* Grothendieck group of  $\text{Perf}(X_n)$  (see Remark 3.7 below. For a definition of the symplectic representation, the reader can refer to [FM], Chapter 6).

The paper is organized as follows. In Section 2, I give some background on the mapping class group, and then work out a convenient presentation of  $\text{PM}(\mathbb{T}_n)$ . The proof of the main result, Theorem 3.6, is contained in Section 3. Theorem 3.6 generalizes previous results in [ST] and in [BK], where the authors considered, respectively, the case of a smooth elliptic curve, and of the nodal cubic in  $\mathbb{P}^2$  (i.e. the case  $n = 1$ ). Equivalences of  $D^b(\text{Coh}(X_n))$  were also investigated in [P]. However, as the author in [P] restricts to a subgroup of equivalences satisfying certain homological conditions, which are violated by the spherical twists I shall consider below, there is essentially no overlap between his work and the present project.

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## 2. THE MAPPING CLASS GROUP OF A PUNCTURED TORUS.

In this section I will briefly review some basic facts about the mapping class group, and then give a presentation of the mapping class group of the punctured torus based on [LP]. Also, it will be useful to spell out some relations between mapping classes which were found by Gervais in [G]. For a comprehensive introduction to the mapping class group I refer the reader to [FM].

Let  $\Sigma = \Sigma_{g,n,b}$  a differentiable, oriented surface of genus  $g$ , with  $n$  marked points, and  $b$  boundary components. The mapping class group of  $\Sigma$ , denoted by  $\text{M}(\Sigma)$ , is the group of isotopy classes of orientation preserving diffeomorphisms of  $\Sigma$ , which send the set of marked points to itself, and restrict to the identity on the boundary components. Note that  $\text{M}(\Sigma_{g,n,b})$

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<sup>1</sup>Note also that the HMS statement in [STZ] involves  $\text{Perf}(X_n)$ , rather than the full derived category of  $X_n$ . However,  $D^b(\text{Coh}(X_n))$  has the same group of auto-equivalences of  $\text{Perf}(X_n)$ . In fact, any equivalence of  $D^b(\text{Coh}(X_n))$  gives, by restriction, an equivalence of  $\text{Perf}(X_n)$ , and it follows from Lemma 3.3 that this assignment is a bijection.

is uniquely determined by the parameters  $g, n, b$ . The *pure* mapping class group of  $\Sigma$  is the subgroup  $\text{PM}(\Sigma) \hookrightarrow \text{M}(\Sigma)$  of mapping classes fixing pointwise the set of marked points. Alternatively,  $\text{PM}(\Sigma)$  can be defined as the subgroup of  $\text{M}(\Sigma)$  generated by Dehn twists along simple closed curves (for the definition of Dehn twist, and a proof of this claim, see Chapter 3 and 4 of [FM]). In making the above definitions, marked points on  $\Sigma$  could be interpreted, equivalently, as punctures, and I shall make use freely of both viewpoints in the following.

A surface  $\Sigma$  with  $n$  punctures and  $b+m$  boundary components can be immersed in a surface with  $n+m$  punctures and  $b$  boundary components (we can trade  $m$  boundary components for  $m$  punctures, by gluing a punctured disc along each boundary component we wish to remove). Further, this immersion induces a map of pure mapping class groups. The details can be found in Section 2 of [LP], together with the following lemma which will be useful later.

**Lemma 2.1.** *Let  $(g, r, m) \notin \{(0, 0, 1), (0, 0, 2)\}$ , then we have the exact sequence*

$$1 \rightarrow \mathbb{Z}^m \rightarrow \text{PM}(\Sigma_{g,n,b+m}) \rightarrow \text{PM}(\Sigma_{g,n+m,b}) \rightarrow 1,$$

where  $\mathbb{Z}^m$  stands for the free abelian group of rank  $m$  generated by the Dehn twists along the  $m$  boundary components we are removing.

Set  $\mathbb{T}_n = \Sigma_{1,n,0}$  and  $\mathbb{T}_{n,m} = \Sigma_{1,n,m}$ . The pure mapping class group  $\text{PM}(\mathbb{T}_n)$  is generated by Dehn twists along  $n+1$  non-separating simple closed curves. In order to fix ideas, it is convenient to choose explicit representatives for this collection of curves. I will mostly follow the notation of [G], to which I refer for further details. Let  $\Lambda = \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$  be the standard integral lattice, let  $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ , and fix a fundamental domain for the action of  $\Lambda$ , say  $[0, 1) \times [0, 1)$ . Choose as set of marked points  $P = \{p_1 = (\frac{1}{n+1}, \frac{1}{2}), \dots, p_n = (\frac{n}{n+1}, \frac{1}{2})\}$ , and identify the index set  $\{1 \dots n\}$  with  $\mathbb{Z}/n$  endowed with the natural cyclic order<sup>2</sup>. A cyclic order allows us to speak unambiguously about ordered triples. If  $i, j, k \in \{1 \dots n\}$  (not necessarily distinct) form an ordered triple, I shall write  $i \preceq j \preceq k$ . If I require  $i, j, k$  to be distinct, I will use the symbol  $\prec$ .

Let  $\alpha$  and  $\beta_i$ ,  $i \in \{1 \dots n\}$  be the following simple closed curves: in the fundamental domain,  $\alpha$  is given by  $[0, 1) \times \{\frac{1}{3}\}$ , and  $\beta_i$  is given by  $\{\frac{i}{n+1} - \frac{1}{2(n+1)}\} \times [0, 1)$ . It will be important to consider also separating curves  $\gamma_{i,j}$  indexed by an ordered pair  $i, j \in \{1 \dots n\}$ . The loop  $\gamma_{i,j}$  can be described as the boundary of a tubular neighborhood of a straight segment  $\sigma$  in  $\mathbb{T}$ , starting at  $p_i$  and ending at  $p_j$ , and such that  $p_k \in \sigma$  if and only if  $i \preceq k \preceq j$ .

If  $\mu$  is a simple closed curve in a differentiable surface  $\Sigma$ , denote  $T_\mu$  the Dehn twist along it. I will consider  $\mathbb{T}_{n-1,1}$  to be the closed subsurface of  $\mathbb{T}_n$  obtained by cutting out a small open disc centered in  $p_n$  (small means that its boundary should not intersect any of the loops described above). It follows from [LP] that both  $\text{PM}(\mathbb{T}_n)$  and  $\text{PM}(\mathbb{T}_{n-1,1})$  are generated by Dehn twists  $T_\alpha$ , and  $T_{\beta_i}$ ,  $i \in \{1 \dots n\}$ . I will refer to this collection of Dehn twists as *Humphrey generators*, in analogy with Humphrey's set of generators for the mapping class group of a compact surface.

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<sup>2</sup>In order to make sense of the successor operator  $\bullet + 1$  on the index set, I will also use the additive structure of  $\mathbb{Z}/n$ .

A presentation of  $\text{PM}(\mathbb{T}_{n-1,1})$  in terms of Humphrey generators can be read off Proposition 3.3 of [LP]. For the reader's convenience I collect it below.

**Proposition 2.2.** *The pure mapping class group  $\text{PM}(\mathbb{T}_{n-1,1})$  is generated by  $T_\alpha$ , and  $T_{\beta_i}$ ,  $i \in \{1 \dots n\}$ , subject to the following relations:*

- (Braid relations) for every  $i, j \in \{1 \dots n\}$ ,  

$$T_{\beta_i} T_{\beta_j} = T_{\beta_j} T_{\beta_i},$$

$$T_\alpha T_{\beta_i} T_\alpha = T_{\beta_i} T_\alpha T_{\beta_i}.$$
- (Commutativity relations) for every  $i, j, k \in \{1 \dots n\}$ ,  $i \prec j \prec k$ ,  

$$T_{\beta_i} (T_\alpha^{-1} T_{\beta_{k+1}}^{-1} T_{\beta_j}^{-1} T_\alpha^{-1} T_{\beta_k} T_\alpha T_{\beta_j} T_{\beta_{k+1}} T_\alpha) = (T_\alpha^{-1} T_{\beta_{k+1}}^{-1} T_{\beta_j}^{-1} T_\alpha^{-1} T_{\beta_k} T_\alpha T_{\beta_j} T_{\beta_{k+1}} T_\alpha) T_{\beta_i}.$$

In order to derive from Proposition 2.2 a presentation for  $\text{PM}(\mathbb{T}_n)$  in terms of Humphrey generators, I will use Lemma 2.1. It follows immediately from the definition of Dehn twist that, for all  $i \in \{1 \dots n\}$ ,  $T_{\gamma_{i,i}} = 1$  in  $\text{PM}(\mathbb{T}_n)$ . In  $\mathbb{T}_{n-1,1}$ , however,  $\gamma_{n,n}$  is isotopic to the boundary component, and therefore  $T_{\gamma_{n,n}}$  defines a non trivial mapping class. Lemma 2.1 assures us that the only extra relation needed to obtain  $\text{PM}(\mathbb{T}_n)$  from the presentation in Proposition 2.2 is precisely  $T_{\gamma_{n,n}} = 1$ . What is left to do is finding an expression for  $T_{\gamma_{n,n}}$  as a product of Humphrey generators. To achieve this, I need to introduce two more ingredients. The first is the following lemma,

**Lemma 2.3.** *Let  $i, j \in \{1 \dots n\}$ , and set*

$$A_{i,j} = T_{\beta_{j+1}} T_\alpha T_{\beta_{i+1}}^{-1} T_{\beta_i} T_\alpha^{-1} T_{\beta_{j+1}}^{-1} T_\alpha T_{\beta_i}^{-1} T_{\beta_{i+1}} T_\alpha^{-1} T_{\beta_{i+1}}^{-1} T_{\beta_i},$$

*then  $T_{\gamma_{i,j}} = A_{i,j} T_{\gamma_{i+1,j}}$ .*

*Proof.* The proof involves standard manipulations in the mapping class group, based on Birman's exact sequence (see [FM], Theorem 4.6). The key point is that  $T_{\gamma_{i,j}} T_{\gamma_{i+1,j}}^{-1}$  lies in the image of Birman's *Push map* (I follow the terminology of [FM]), and this allows to reduce computations in the mapping class group to computations in the fundamental group of the torus with  $n - 1$  punctures.  $\square$

The second ingredient is given by a family of relations in the mapping class group, introduced by Gervais in [G] as *star relations*.

**Proposition 2.4.** *Let  $i, j, k \in \{1 \dots n\}$ , such that they are not all equal, and  $i \preceq j \preceq k$ . Then*

$$(T_\alpha T_{\beta_i} T_{\beta_j} T_{\beta_k})^3 = T_{\gamma_{i,j}} T_{\gamma_{j,k}} T_{\gamma_{k,i}}.$$

*Proof.* See Theorem 1 in [G].  $\square$

In fact, one can remove the condition that  $i, j, k$  be distinct, and set  $i = j = k$ . This yields the following 'degenerate' star relations which are especially relevant for my purpose, and follow immediately from Proposition 2.4. For all  $i \in \{1 \dots n\}$ ,

$$(T_\alpha T_{\beta_i})^6 = T_{\gamma_{i,i-1}}.$$

Let's fix  $i \in \{1 \dots n\}$ , say  $i = 1$ , then the degenerate star identity for  $i = 1$  combined with an iterated application of Lemma 2.3 (from which we import the notation), gives the

formula

$$(T_\alpha T_{\beta_1})^6 = (A_{1,n} A_{2,n} \dots A_{n-1,n}) T_{\gamma_{n,n}}.$$

Since the  $A_{i,j}$ -s are defined as a product of  $T_\alpha$  and  $T_{\beta_i}$ -s, this yields the sought after expression of  $T_{\gamma_{n,n}}$  in terms of Humphrey generators, and thus the following proposition.

**Proposition 2.5.** *The pure mapping class group  $\text{PM}(\mathbb{T}_n)$  is generated by  $T_\alpha$ , and  $T_{\beta_i}$ ,  $i \in \{1 \dots n\}$ , subject to the following relations:*

- *Braid relations and Commutativity relations (see Proposition 2.2)*
- *(G-relation)  $(T_\alpha T_{\beta_1})^6 = A_{1,n} A_{2,n} \dots A_{n-1,n}$ .*

Note that the  $G$ -relation depends on the degenerate star identity for  $i = 1$ . However in  $\text{PM}(\mathbb{T}_n)$  one would have more generally, for any  $i \in \{1 \dots n\}$ ,

$$(T_\alpha T_{\beta_i})^6 = A_{i,i+n-1} A_{i+1,i+n-1} \dots A_{i+n-2,i+n-1}.$$

Further, because of the cyclic symmetry of the problem, one could replace the  $G$ -relation of Proposition 2.5 with any one of the identities above, and still get a presentation of  $\text{PM}(\mathbb{T}_n)$ . Thus, modulo braid and commutativity relations, each one of these identities implies all the others. In fact, a slightly stronger claim is true:

**Remark 2.6.** Let  $i \in \{1 \dots n\}$ , then the following identity holds in  $\text{PM}(\mathbb{T}_{n-1,1})$ :

$$(T_\alpha T_{\beta_1})^6 (T_\alpha T_{\beta_i})^{-6} = (A_{1,n} A_{2,n} \dots A_{n-1,n}) (A_{i,i+n-1} A_{i+1,i+n-1} \dots A_{i+n-2,i+n-1})^{-1}.$$

In the next section, when I will describe the action of  $\text{PM}(\mathbb{T}_n)$  on  $D^b(\text{Coh}(X_n))$ , it will be convenient to break up the argument in the cases  $n \leq 2$ , and  $n > 2$ . I will need the following alternative presentation of  $\text{PM}(\mathbb{T}_2)$ , which was found in [PS], and can be easily seen to be compatible with the presentation in Proposition 2.5.

**Remark 2.7.** The pure mapping class group  $\text{PM}(\mathbb{T}_2)$  is generated by  $T_\alpha$ ,  $T_{\beta_1}$  and  $T_{\beta_2}$ , with relations

- Braid relations as in Proposition 2.5,
- $(G_2\text{-relation}) (T_{\beta_1} T_\alpha T_{\beta_2})^4 = 1$ .

Note that the  $G_2$ -relation as well follows from Gervais' star relations.

### 3. THE ACTION OF $\widetilde{\text{PM}}(\mathbb{T}_n)$ ON $D^b \text{Coh}(X_n)$ .

Let  $X_n$  be a cycle of  $n$  projective lines over a field  $\kappa$ . That is,  $X_n$  is a connected reduced curve with  $n$  nodal singularities, such that its normalization  $\tilde{X}_n \xrightarrow{\pi} X$  is a disjoint union of  $n$  projective lines  $D_1, \dots, D_n$ , with the property that the pre-image along  $\pi$  of the singular set intersects each  $D_i$  in exactly two points. Following the discussion in Section 1 of [ST], the group acting on  $D^b \text{Coh}(X_n)$  is going to be a suitable central extension of  $\text{PM}(\mathbb{T}_n)$ , whose elements should be viewed as *graded* symplectic automorphisms of the mirror of  $X_n$ , i.e. the torus with  $n$  punctures.

**Definition 3.1.** Define  $\widetilde{\text{PM}}(\mathbb{T}_n)$  as the  $\mathbb{Z}$ -central extension of  $\text{PM}(\mathbb{T}_n)$ ,

$$0 \rightarrow \mathbb{Z} \rightarrow \widetilde{\text{PM}}(\mathbb{T}_n) \rightarrow \text{PM}(\mathbb{T}_n) \rightarrow 1,$$

generated by  $T_\alpha$ ,  $T_{\beta_i}$   $i \in \{1 \dots n\}$ , and a central element  $t$ , subject to the following relations:

- Braid relations and Commutativity relations, as in Proposition 2.5,
- ( $\tilde{G}$ -relation)  $(T_\alpha T_{\beta_1})^6 (A_{1,n} A_{2,n} \dots A_{n-1,n})^{-1} = t^2$ .

**Remark 3.2.** By lifting the  $G_2$ -relation of Remark 2.7 to the central extension, one can give an alternative presentation of  $\widetilde{\text{PM}}(\mathbb{T}_2)$  in which the  $\tilde{G}$ -relation of Definition 3.1 is replaced by the following,

- ( $\tilde{G}_2$ -relation)  $(T_{\beta_1} T_\alpha T_{\beta_2})^4 = t^2$ .

The theory of spherical objects was introduced by Seidel and Thomas in [ST]. Given a triangulated category  $C$ , under mild assumptions, to any object  $\mathcal{E}$  in  $C$  such that  $\text{Hom}^*(\mathcal{E}, \mathcal{E})$  is isomorphic to the cohomology of the  $n$ -sphere (i.e. a *spherical object*), one can associate an autoequivalence, called *twist*,  $T_{\mathcal{E}} : C \rightarrow C$ .

Let  $x_1 \dots x_n \in X_n$  be (closed) smooth points, such that  $x_i$  lies on the  $i$ -th irreducible component of  $X_n$ . It is easy to see that the sheaves  $\mathcal{O} = \mathcal{O}_{X_n}$ ,  $\kappa(x_i)$   $i \in \{1 \dots n\}$  in  $D^b(\text{Coh}(X_n))$  are 1-spherical, and therefore determine twist functors  $T_{\mathcal{O}}, T_{\kappa(x_i)}$ . These equivalences, together with the *shift* functor, will define the action of  $\widetilde{\text{PM}}(\mathbb{T}_n)$  on  $D^b(\text{Coh}(X_n))$ . The main reference for the computations below are [ST] and [BK]. In [BK] the reader can find a detailed treatment of the case  $n = 1$ , while in [ST] Seidel and Thomas discuss the smooth case, i.e. the action of the mapping class group of a torus with no marked points on the derived category of a smooth elliptic curve.

The following lemma will be extremely useful for computations.

**Lemma 3.3.** *Let  $F : D^b(\text{Coh}(X_n)) \rightarrow D^b(\text{Coh}(X_n))$  be an auto-equivalence of triangulated categories. If*

- $F(\mathcal{O}) \cong \mathcal{O}$ , and
- for all  $i \in \{1 \dots n\}$ ,  $F(\kappa(x_i)) \cong \kappa(x_i)$ ,

*then there exists an isomorphism  $f : X_n \rightarrow X_n$ , such that  $F$  is naturally equivalent to  $f^* : D^b(\text{Coh}(X_n)) \rightarrow D^b(\text{Coh}(X_n))$ .*

*Proof.* Note that  $X_n$  is projective, as  $X_1$  is isomorphic to a nodal cubic curve in  $\mathbb{P}^2$ ,  $X_2$  can be embedded as the union of a line and a quadric in  $\mathbb{P}^2$ , and, if  $n \geq 3$ ,  $X_n$  can be embedded as a union of  $n$  linear subspaces in  $\mathbb{P}^{n-1}$ . Consider the line bundle  $\mathcal{L} = \mathcal{O}(x_1 + \dots + x_n)$  over  $X_n$ .  $\mathcal{L}$  is ample (and very ample for  $n \geq 3$ ). Since  $F$  preserves  $\mathcal{O}$  and  $\kappa(x_i)$ , it is easy to see that  $F(\mathcal{L}^{\otimes m}) \cong \mathcal{L}^{\otimes m}$  for all  $m \in \mathbb{Z}$ . In fact,  $\mathcal{L}^{-1}$  is isomorphic to the kernel of any surjective morphism of sheaves  $p : \mathcal{O} \rightarrow \bigoplus_{i=1}^{i=n} \kappa(x_i)$ .  $F(\mathcal{L}^{-1})$  is therefore isomorphic to the (co-)cone of the map

$$F(\mathcal{O}) (\cong \mathcal{O}) \xrightarrow{F(p)} F\left(\bigoplus_{i=1}^{i=n} \kappa(x_i)\right) (\cong \bigoplus_{i=1}^{i=n} \kappa(x_i)),$$

where  $F(p)$  must be surjective. It follows that  $F(\mathcal{L}^{-1}) \cong \mathcal{L}^{-1}$ . Similarly  $\mathcal{L}$  is isomorphic to the cone of any morphism in  $\text{Hom}^1(\bigoplus_{i=1}^{i=n} \kappa(x_i), \mathcal{O})$  corresponding, under Serre duality, to a surjective morphism  $p$  as above, and thus  $F(\mathcal{L}) \cong \mathcal{L}$ . Analogous arguments can be made for all the tensor powers of  $\mathcal{L}$ .

From here, in order to prove the claim, is sufficient to mimic the proof of Theorem 3.1 of [BO] (see also Proposition 6.18 in [Ba], in which the argument from [BO] is applied, as here, in



the context of singular algebraic varieties). A brief summary of the argument goes as follows. Note first that the functor  $F$  induces a graded automorphism of the homogeneous coordinate algebra  $\bigoplus_{m=0}^{m=\infty} H^0(\mathcal{L}^{\otimes m})$ , which, up to rescaling, must be given by the pull-back along an automorphism  $f : X_n \rightarrow X_n$ . Call  $C$  the full linear sub-category of  $D^b(\text{Coh}(X_n))$  having as objects  $\{\mathcal{L}^{\otimes m}\}_{m \in \mathbb{Z}}$ . It is easy to define a natural equivalence between the restrictions to  $C$  of  $F$  and  $f^*$ . Since  $X_n$  is projective, and  $\mathcal{L}$  is ample,  $\{\mathcal{L}^{\otimes m}\}_{m \in \mathbb{Z}}$  form an ample sequence in the sense of [BO] (for a proof of this, see Proposition 3.18 of [Huy]). The claim then follows from Proposition A.3 of [BO], which implies that the natural equivalence  $F \cong f^*$  over  $C$  can be extended to the full derived category  $D^b(\text{Coh}(X_n))$ .  $\square$

**Remark 3.4.** Note that if  $f : X_n \rightarrow X_n$  is an automorphism such that  $f(x_i) = x_i$ , and  $n \geq 3$ , then  $f$  is the identity. If  $n \leq 2$ ,  $f$  may be non-trivial and act as a (non-trivial) permutation on the pre-image of the singular locus in the normalization. However it is immediate to see that  $f$  is an involution, i.e.  $f^2 = \text{Id}$ .

**Lemma 3.5.** *Let  $x \in X_n$  be a smooth point, then*

- $T_{\kappa(x)} \cong - \otimes \mathcal{O}(x)$ ,
- $T_{\mathcal{O}}(\kappa(x)) \cong \mathcal{O}(-x)[1]$ ,
- $T_{\mathcal{O}}(\mathcal{O}(x)) \cong \kappa(x)$ ,
- $T_{\mathcal{O}}(\mathcal{O}) \cong \mathcal{O}$ .

*Proof.* The first isomorphism is proved in [ST], Section 3.d. For the other isomorphisms, see Lemma 2.13 in [BK].  $\square$

I am now ready to state the main theorem of this paper.

**Theorem 3.6.** *The assignment*

- *for all  $i \in \{1 \dots n\}$ ,  $T_{\beta_i} \mapsto T_{\kappa(x_i)}$ ,*
- *$T_{\alpha} \mapsto T_{\mathcal{O}}$ , and*
- *$t \mapsto [1]$ ,*

*defines a weak action of  $\widetilde{\text{PM}}(\mathbb{T}_n)$  on  $D^b(\text{Coh}(X_n))$ .*

Following [ST], by weak action I mean that this assignment defines a homomorphism between  $\widetilde{\text{PM}}(\mathbb{T}_n)$ , and the group of autoequivalences of  $D^b(\text{Coh}(X_n))$  modulo natural isomorphism of functors. The action defined in Theorem 3.6 depends on the choice of  $x_1, \dots, x_n$ . However, the action is unique up to conjugation. Note that there is a natural  $(\mathbb{C}^*)^n$ -action on  $X_n$ , with the property that the  $i$ -th copy of  $\mathbb{C}^*$  acts by multiplication on the  $i$ -th component of  $X_n$ . Let  $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$ , and let  $m_{\lambda} : X_n \rightarrow X_n$  be the associated automorphism. Then one can show that, for all  $i \in \{1, \dots, n\}$ ,  $(m_{\lambda}^*)T_{\kappa(x_i)}(m_{\lambda}^*)^{-1} = T_{\kappa(\lambda_i x_i)}$ , and  $(m_{\lambda}^*)T_{\mathcal{O}}(m_{\lambda}^*)^{-1} = T_{\mathcal{O}}$ .

*Proof of Theorem 3.6.* I will show that Theorem 3.6 gives a well-defined homomorphism, by checking that the relations in Definition 3.1 hold.

*Braid relations.* For all  $i, j \in \{1, \dots, n\}$ ,  $\mathcal{O}$ ,  $\kappa(x_i)$  and  $\kappa(x_j)$  form an  $A_2$ -configuration, in the language of [ST]. The fact that such a collection of spherical twists satisfies the braid relations is proved in Proposition 2.13 of [ST].

*Commutativity relations.* By Lemma 2.11 of [ST], if  $\mathcal{E}_1, \mathcal{E}_2$  are spherical objects, then  $T_{\mathcal{E}_2}T_{\mathcal{E}_1}T_{\mathcal{E}_2}^{-1} \cong T_{T_{\mathcal{E}_2}(\mathcal{E}_1)}$ . It follows that, in order to prove the Commutativity relations, is

sufficient to show that, for every  $i, j, k \in \{1 \dots n\}$ ,  $i \prec j \prec k$ ,

$$\begin{aligned} T_{\mathcal{O}}^{-1} T_{\kappa(x_{k+1})}^{-1} T_{\kappa(x_j)}^{-1} (T_{\mathcal{O}}^{-1} T_{\kappa(x_k)} T_{\mathcal{O}}) T_{\kappa(x_j)} T_{\kappa(x_{k+1})} T_{\mathcal{O}}(\kappa(x_i)) &\cong \kappa(x_i) \Leftrightarrow \\ T_{\mathcal{O}}^{-1} T_{\kappa(x_{k+1})}^{-1} T_{\kappa(x_j)}^{-1} (T_{\kappa(x_k)} T_{\mathcal{O}} T_{\kappa(x_k)}^{-1}) T_{\kappa(x_j)} T_{\kappa(x_{k+1})} T_{\mathcal{O}}(\kappa(x_i)) &\cong \kappa(x_i) \Leftrightarrow^3 \\ T_{\mathcal{O}} T_{\kappa(x_k)}^{-1} T_{\kappa(x_j)} T_{\kappa(x_{k+1})} T_{\mathcal{O}}(\kappa(x_i)) &\cong T_{\kappa(x_k)}^{-1} T_{\kappa(x_j)} T_{\kappa(x_{k+1})} T_{\mathcal{O}}(\kappa(x_i)). \end{aligned}$$

By Lemma 3.5  $T_{\kappa(x_k)}^{-1} T_{\kappa(x_j)} T_{\kappa(x_{k+1})} T_{\mathcal{O}}(\kappa(x_i)) \cong \mathcal{O}(-x_i + x_j - x_k + x_{k+1})[1]$ . Thus, I need to show that  $T_{\mathcal{O}}(\mathcal{O}(-x_i + x_j - x_k + x_{k+1})) \cong \mathcal{O}(-x_i + x_j - x_k + x_{k+1})$ . Proposition 2.12 of [ST] states that if  $\mathcal{E}_1, \mathcal{E}_2$  are spherical objects such that  $\text{Hom}^i(\mathcal{E}_1, \mathcal{E}_2) = 0$  for all  $i$ , then  $T_{\mathcal{E}_1}(\mathcal{E}_2) \cong \mathcal{E}_2$ . The Commutativity relations reduce therefore to the claim: for all  $i, j, k \in \{1 \dots n\}$ ,  $i \prec j \prec k$ ,

$$H^0(\mathcal{O}(-x_i + x_j - x_k + x_{k+1})) = H^1(\mathcal{O}(-x_i + x_j - x_k + x_{k+1})) = 0.$$

This follows from Theorem 2.2 of [DGK], which gives a general formula for computing the cohomology groups of indecomposable vector bundles over a cycle of projective lines.

*$\tilde{G}$ -relation.* Assume first that  $n \geq 3$ . I will handle separately the case  $n = 2$ , for which I will use of the alternative  $\tilde{G}_2$ -relation of Remark 3.2 (for the case  $n = 1$ , the reader should refer to [BK]). Let  $i, j \in \{1 \dots n\}$ , and define

$$E_{i,j} = T_{\kappa(x_{j+1})} T_{\mathcal{O}} T_{\kappa(x_{i+1})}^{-1} T_{\kappa(x_i)} T_{\mathcal{O}}^{-1} T_{\kappa(x_{j+1})}^{-1} T_{\mathcal{O}} T_{\kappa(x_i)}^{-1} T_{\kappa(x_{i+1})} T_{\mathcal{O}}^{-1} T_{\kappa(x_{i+1})}^{-1} T_{\kappa(x_i)}.$$

I need to prove that  $(T_{\mathcal{O}} T_{\kappa(x_1)})^6 \cong (E_{1,n} E_{2,n} \dots E_{n-1,n})[2]$ . After Lemma 3.3 and Remark 3.4, it is sufficient to check the  $\tilde{G}$ -relation on  $\mathcal{O}$ , and  $\kappa(x_i)$  for all  $i \in \{1 \dots n\}$ . In fact, in view of Remark 2.6, it is enough to evaluate the  $\tilde{G}$ -relation on  $\mathcal{O}$  and  $\kappa(x_1)$ , as, for any  $k \in \{1 \dots n\}$ ,

$$\begin{aligned} (T_{\mathcal{O}} T_{\kappa(x_1)})^6(\kappa(x_k)) &\cong (E_{1,n} E_{2,n} \dots E_{n-1,n})[2](\kappa(x_k)) \Leftrightarrow \\ (T_{\mathcal{O}} T_{\kappa(x_k)})^6(\kappa(x_k)) &\cong (E_{i,i+n-1} E_{i+1,i+n-1} \dots E_{i+n-2,i+n-1})[2](\kappa(x_k)), \end{aligned}$$

and, by the cyclic symmetry of the problem, the latter identity is proved in exactly the same way as the claim that the  $\tilde{G}$ -relation holds for  $\kappa(x_1)$ .

•  $\tilde{G}$ -relation on  $\mathcal{O}$ . Simply by keeping track of the isomorphisms collected in Lemma 3.5, one can see that

$$(T_{\mathcal{O}} T_{\kappa(x_1)})^6(\mathcal{O}) \cong (T_{\mathcal{O}} T_{\kappa(x_1)})^4(\mathcal{O}(-x_1)[1]) \cong (T_{\mathcal{O}} T_{\kappa(x_1)})^2(\kappa(x_1)[1]) \cong \mathcal{O}[2].$$

On the other hand, I will show that, for all  $i, j \in \{1, \dots, n\}$ ,  $E_{i,j}(\mathcal{O}) \cong \mathcal{O}$ , and therefore

$$(E_{1,n} E_{2,n} \dots E_{n-1,n})[2](\mathcal{O}) \cong \mathcal{O}[2],$$

as expected. Note first that if  $x, y \in X_n$  are smooth points lying on different connected components, then  $T_{\mathcal{O}}(\mathcal{O}(x-y)) \cong \mathcal{O}(x-y)$ . This again follows from Theorem 2.2 of [DGK], but can also be checked directly using the braid relations. Using this isomorphism, and Lemma 3.5, it is easy to check that

$$\begin{aligned} E_{i,j}(\mathcal{O}) &= T_{\kappa(x_{j+1})} T_{\mathcal{O}} T_{\kappa(x_{i+1})}^{-1} T_{\kappa(x_i)} T_{\mathcal{O}}^{-1} T_{\kappa(x_{j+1})}^{-1} T_{\mathcal{O}} T_{\kappa(x_i)}^{-1} T_{\kappa(x_{i+1})} T_{\mathcal{O}}^{-1} T_{\kappa(x_{i+1})}^{-1} T_{\kappa(x_i)}(\mathcal{O}) \cong \\ &T_{\kappa(x_{j+1})} T_{\mathcal{O}} T_{\kappa(x_{i+1})}^{-1} T_{\kappa(x_i)} T_{\mathcal{O}}^{-1} T_{\kappa(x_{j+1})}^{-1} T_{\mathcal{O}} T_{\kappa(x_i)}^{-1} T_{\kappa(x_{i+1})}(\mathcal{O}(x_i - x_{i+1})) \cong \end{aligned}$$

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<sup>3</sup>The isomorphism  $T_{\mathcal{O}}^{-1} T_{\kappa(x_k)} T_{\mathcal{O}} \cong T_{\kappa(x_k)} T_{\mathcal{O}} T_{\kappa(x_k)}^{-1}$  follows immediately from the braid relations.



$$T_{\kappa(x_{j+1})}T_{\mathcal{O}}T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_{j+1})}^{-1}(\mathcal{O}) \cong$$

$$T_{\kappa(x_{j+1})}T_{\mathcal{O}}T_{\kappa(x_{i+1})}^{-1}(\kappa(x_{j+1})[-1]) \cong \mathcal{O}.$$

- $\tilde{G}$ -relation on  $\kappa(x_1)$ . As before, it is enough to apply Lemma 3.5 to see that

$$(T_{\mathcal{O}}T_{\kappa(x_1)})^6(\kappa(x_1)) \cong (T_{\mathcal{O}}T_{\kappa(x_1)})^4(\mathcal{O}[1]) \cong (T_{\mathcal{O}}T_{\kappa(x_1)})^2(\mathcal{O}(-x_1)[2]) \cong \kappa(x_1)[2].$$

Further, for all  $i \in \{1, \dots, n-1\}$ ,  $E_{i,n}(\kappa(x_1)) \cong \kappa(x_1)$ . In fact,

$$E_{i,n}(\kappa(x_1)) = T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_1)}^{-1}T_{\mathcal{O}}T_{\kappa(x_i)}^{-1}T_{\kappa(x_{i+1})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_i)}(\kappa(x_1)) \cong$$

$$T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_1)}^{-1}T_{\mathcal{O}}T_{\kappa(x_i)}^{-1}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)).$$

Now,

$$T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_1)}^{-1}T_{\mathcal{O}}T_{\kappa(x_i)}^{-1}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)) \cong \kappa(x_1) \Leftrightarrow$$

$$T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_i)}(T_{\mathcal{O}}^{-1}T_{\kappa(x_1)}^{-1}T_{\mathcal{O}})T_{\kappa(x_i)}^{-1}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)) \cong \mathcal{O}(x_1) \Leftrightarrow^4$$

$$T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_i)}(T_{\kappa(x_1)}T_{\mathcal{O}}^{-1}T_{\kappa(x_1)}^{-1})T_{\kappa(x_i)}^{-1}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)) \cong \mathcal{O}(x_1) \Leftrightarrow$$

$$T_{\mathcal{O}}^{-1}T_{\kappa(x_1)}^{-1}T_{\kappa(x_i)}^{-1}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)) \cong \mathcal{O}(x_{i+1} - x_i) \Leftrightarrow$$

$$\mathcal{O}(x_{i+1} - x_i) \cong T_{\mathcal{O}}(\mathcal{O}(x_{i+1} - x_i)).$$

As I pointed out above, this last isomorphism can be proved using the braid relations. Thus

$$(E_{1,n}E_{2,n} \dots E_{n-1,n})[2](\kappa(x_1)) \cong \kappa(x_1)[2],$$

and this concludes the proof of Theorem 3.6 for the case  $n \geq 3$ .

*The case  $n = 2$ .* Note that there are isomorphisms

- $(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^2(\mathcal{O}) \cong \mathcal{O}[1]$ , and
- $(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^2(\kappa(x_1)) \cong \kappa(x_2)[1]$ ,  $(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^2(\kappa(x_2)) \cong \kappa(x_1)[1]$ .

Let's check this for  $\kappa(x_1)$ :

$$(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})(\kappa(x_1)) \cong (T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})(\mathcal{O}[1]) \cong \kappa(x_2)[1].$$

Consider an involution  $\sigma : X_2 \rightarrow X_2$  such that  $\sigma(x_1) = x_2$ , and  $\sigma(x_2) = x_1$ . It follows from Lemma 3.4 that there is an isomorphism  $f : X_2 \rightarrow X_2$ , and a natural equivalence  $(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^2 \cong f^* \sigma^*[1]$ . As  $\sigma$  and  $f$  commute, by taking the square of this natural equivalence, one gets

$$(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^4 \cong (f^* \sigma^*[1])(f^* \sigma^*[1]) \cong (f^*)^2(\sigma^*)^2[2] \cong [2].$$

In view of Remark 3.2, this implies that the action of  $\widetilde{\text{PM}}(\mathbb{T}_2)$  on  $D^b(\text{Coh}(X_2))$  is well defined, and proves the case  $n = 2$  of Theorem 3.6.  $\square$

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<sup>4</sup>The isomorphism  $T_{\mathcal{O}}^{-1}T_{\kappa(x_1)}^{-1}T_{\mathcal{O}} \cong T_{\kappa(x_1)}T_{\mathcal{O}}^{-1}T_{\kappa(x_1)}^{-1}$  is an easy consequence of the braid relations.

**Remark 3.7.** It follows from results in Appendix D of [B], that the action defined by Theorem 3.6 is, in an appropriate sense, a categorification of the symplectic representation of the mapping class group. Denote  $K^{num}(X_n)$  the quotient of  $K_0(\mathcal{P}erf(X_n))$  by the radical of the Euler form (see Appendix D of [B] for further details). The Euler form induces a non-degenerate skew-symmetric form on  $K^{num}(X_n)$ , and there is an isomorphism of symplectic lattices  $K^{num}(X_n) \cong H_1(\mathbb{T}_n, \mathbb{Z}) (\cong \mathbb{Z}^{n+1})$  (here,  $\mathbb{T}_n$  denotes the torus with the  $n$  marked points removed). Note that the induced action of  $\tilde{PM}(\mathbb{T}_n)$  on  $K^{num}(X_n)$  factors through  $PM(\mathbb{T}_n) \oplus \mathbb{Z}_2$ . Bodnarchuk’s computations imply that the resulting action of  $PM(\mathbb{T}_n)$  on  $K^{num}(X_n)$  is isomorphic to the standard symplectic representation of  $PM(\mathbb{T}_n)$  over  $H_1(\mathbb{T}_n, \mathbb{Z})$ .

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NICOLÒ SIBILLA, DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, 2033 SHERIDAN ROAD, EVANSTON, IL 60208

*E-mail address:* [sibilla@math.northwestern.edu](mailto:sibilla@math.northwestern.edu)